ANALYSIS OF THE NONLINEAR DEFORMATION OF COMPOSITES WITH ALLOWANCE FOR FINITE ROTATIONS OF STRUCTURAL ELEMENTS

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New high-modulus non-fabric-based synthetic materials are now being developed and widely introduced. These composites have a fibrous-reticular structure (close to rhomboid) and are impregnated with a polymer binder. As micrographs have shown, the binder ends up mainly at the nodes network-like structure of the material. Thus, a certain volume is occupied by pores located a regular distance apart. As opposed to "hard" composites (reinforced by fibers made of glass, boron, carbon, etc.), we will refer to these materials as "soft" composites; they are characterized by high levels of elasticity and plasticity and, at sufficiently small loads, they manifest appreciably nonlinear properties. The possession of such properties is connected with changes in the structure of the material. In particular, substructural elements are rotated relative to each other by a finite amount. There is also a significant change in the porosity of the material. Allowance for these effects makes it possible to determine the transverse strains which occur under uniaxial loading. Thus, in constructing physical relations for these composites, we used the structural approach in [1]. This approach makes it possible to consider the following: the unit volume content of fibers and binder; porosity; the character of reinforcement of the composite; nonlinear properties of the substructural elements; the anisotropy of these properties (for the binder).

Many authors have used mechanical models [2-5] to describe the behavior of various materials. For example, Rabotnov [2] and Askadskii and Matveev [4] used different rheological models in which the elements extended in only one direction. The study [3] used the rod model to examine the limiting state of hard composites reinforced in two directions. An analysis of the effect of the angle between structural rheological elements of the model was made in [4] in an investigation of the relaxation-time spectrum of polymers. Along with the limited scope of the inquiry [4], the approach used here precluded consideration of the structure of the composite and of transverse strains occurring during uniaxial loading. These strains are significant in the deformation of soft composites.

We will use the following model to construct the equations of state of the materials we are examining. We will assume that the fiber composite in our study consists of repeating rhomboid elements of thickness h in the direction of the $O\zeta$ axis (Fig. 1). The fibers of the composite are misoriented by the angle 2α and form elementary rhomboid-cells. The binder is located at the nodes of these cells (where binder is absent at a node, friction develops between the interwoven fibers). The fibers in the weave are in an equilibrium state, and the material is capable of retaining its shape and dimensions in the absence of an external load. When such a model is loaded, the elementary rhomboid-cells are deformed due to a change in the angle 2α between the fibers, as well as due to deformation of the fibers themselves and the binder.

The mechanical behavior of the binder at the nodes of a cell will be modeled by means of elements AC and BD (see Fig. 1). These elements characterize the deformation of the binder at the nodes B, D and A, C, respectively. Here, for the sake of simplicity and definiteness, we assume that the elements BD and AC have a rectangular cross section of thickness h_x, h_y and width b_x, b_y. These quantities are determined from the conditions of equality of the volume of the binder at nodes A, C and B, D.

For the composites we are examining, the type of deformation that it most interesting from a practical viewpoint is tension along the axes of structural symmetry. In this case, considering the structural features of the material (its regularity), we will assume that

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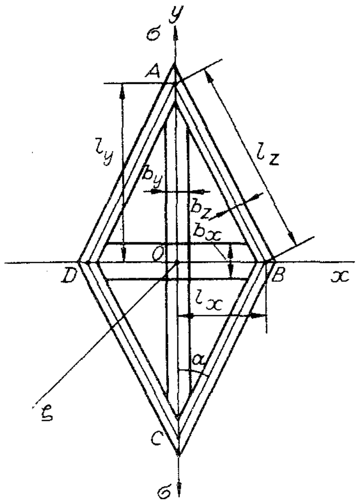

Fig. 1

forces are transmitted only through the nodes and that all of the elements are connected by hinges at these sites.

Let stresses σ be applied at cell nodes A and C (see Fig. 1). The geometric parameters of the cell and the mechanical properties of its elements are such that the coordinate axes Ox and Oy are axes of symmetry in both the unloaded state and during deformation.

We introduce the concept of relative unit volume content for each element of the composite. This quantity is the ratio of the volume of the material of the substructural element to the total volume of the material of the cell. Thus, for fibers (i.e., elements AB, BC, CD, and DA)

$$\Omega_z = 4h_z b_z l_z / V^*, \tag{1}$$

for the binder in the directions Ox and Oy (i.e., the elements BD and AC)

$$\Omega_x = 2h_x b_x l_x / V^*, \quad \Omega_y = 2h_y b_y l_y / V^*, \tag{2}$$

where $V^* = 4h_z b_z \ell_z + 2h_x b_x \ell_x + 2h_y b_y \ell_y$; $h_z = h$, b_z, ℓ_z are the thickness, width, and length of the reinforcing element AB (BC, CD, DA); without loss of generality, for the sake of simplicity and definiteness we assume that the fibers have a rectangular cross section; $2\ell_x$ and $2\ell_y$ are the lengths of elements BD and AC; here, in the expression for V^* the first term corresponds to the volume of the fibers in the cell, while the second and third terms correspond to the volume of the binder at nodes A, C and B, D which are modeled by rods BD and AC. We find from Eqs. (1) and (2) that $\Omega_x + \Omega_y + \Omega_z = 1$ and that the relative unit volume content of binder $\Omega_c = \Omega_x + \Omega_y$.

When a unit cell is tensioned by a stress σ applied at nodes A and C (see Fig. 1), we obtain the following equations of static equilibrium: at node A

$$\sigma(h_y b_y + 2h_z b_z / \cos\alpha) = 2\sigma_z h_z b_z \cos\alpha + \sigma_y h_y b_y; \tag{3}$$

at node B

$$\sigma_x h_x b_x + 2\sigma_z h_z b_z \sin\alpha = 0 \tag{4}$$

(σ_x, σ_y, σ_z are the stresses in elements BD, AC, AB (BC, CD, DA), respectively).

Considering relations (1)-(2) and $\cos\alpha = \ell_y/\ell_z$ and performing certain transformations, we find the following equilibrium equations from (3) and (4):

$$\sigma(1 - \Omega_x) = \sigma_z \Omega_z \cos^2\alpha + \sigma_y \Omega_y, \quad \sigma_x \Omega_x + \sigma_z \Omega_z \sin^2\alpha = 0. \tag{5}$$

We use ε_x and ε_y to denote the strains of the cell in the Ox and Oy directions. These strains coincide with the strains of elements BD and AC (by virtue of the strain compati

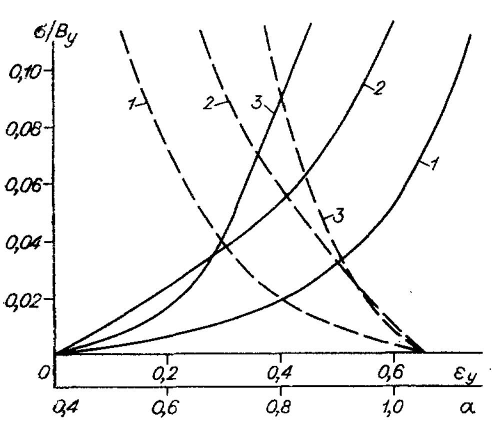

Fig. 2

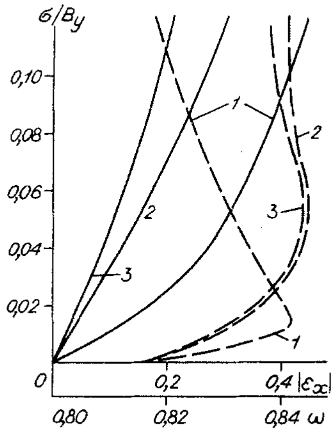

Fig. 3

bility conditions for all elements entering into the nodes A, B, C, D). The strain in the fibers, i.e., in elements AB, BC, CD, and DA will be designated through ε_z. As usual, these strains are determined in the following manner:

$$\varepsilon_x = (l_x - l_{x,0})/l_{x,0},\ \varepsilon_y = (l_y - l_{y,0})/l_{y,0},\\ \varepsilon_z = (l_z - l_{z,0})/l_{z,0}. \tag{6}$$

Here and below, all quantities with the subscript 0 correspond to the initial state before loading, while those without this subscript represent running values during deformation.

We obtain the strain compatibility conditions for all elements of the composite if we examine the triangle AOB (see Fig. 1):

$$l_z^2 = l_x^2 + l_y^2 \text{ and } \mathrm{tg}\,\alpha = l_x / l_y \quad \forall\sigma \geqslant 0,$$

or, taking (6) into account, we obtain

$$(1 + \varepsilon_z)^2 = (1 + \varepsilon_x)^2 \sin^2\alpha_0 + (1 + \varepsilon_y)^2 \cos^2\alpha_0,\\ (1 + \varepsilon_y)\,\mathrm{tg}\,\alpha = (1 + \varepsilon_x)\,\mathrm{tg}\,\alpha_0 \quad \forall\sigma \geqslant 0. \tag{7}$$

In order to close system (5), (7), we need to formulate the equations of state for each element of the cell. For nonlinearly elastic behavior by the elements of the composite, we have

$$\sigma_x = B_x\varepsilon_x|\varepsilon_x|^{n_x-1},\ \sigma_y = B_y\varepsilon_y|\varepsilon_y|^{n_y-1},\ \sigma_z = B_z\varepsilon_z|\varepsilon_z|^{n_z-1}, \tag{8}$$

where B_x, B_y, B_z, n_x, n_y, n_z are the empirical constants of the materials of the binder and fibers. The case $n_x = n_y = n_z = 1$ corresponds to Hooke's law: if $B_x \neq B_y$ or $n_x \neq n_y$, then we can consider the difference in the properties of the binder during deformation in the directions Ox and Oy. It should be noted that, instead of (8), we can use other equations of state for the elements of the composite. For example, we can use relations from linear viscoelasticity [2].

Thus, system (5), (7), (8) completely describes the mechanical behavior of the composite during uniaxial loading. The proposed mathematical model of the composite considers its structure and the difference in the mechanical properties of the binder and fibers, and it determines transverse strain as well as deformation in the direction of the external force. The equations obtained here also make it possible to determine the change in the structure of the composite during loading. For example, the second relation of (7) can be used to determine the change in the angle of reinforcement α during deformation.

Let us introduce the concept of the porosity of the material ω as the ratio of the volume of the pores V to the volume of the entire unit cell V_e:

$$\omega = V/V_e = 1 - V^*/V_e. \tag{9}$$

Here, $V_e = 2\ell_x\ell_y h + 2\ell_z b_z h$. In this case, the first term in the expression for V_e corresponds to the volume of rhomboid-cell ABCD (see Fig. 1), while the second term corresponds to that part of the volume of elements AB, BC, CD, and DA that extends beyond the boundary of the rhomboid ABCD.

For the sake of simplicity and determinateness, we will examine the case when the materials of the binder and fibers are incompressible, the thicknesses of all elements of the cell are the same ($h_x = h_y = h$), and the transverse strain in the direction of the $O\zeta$ axis of the cell can be ignored compared to ε_x, ε_y. Then after some simple transformations we find from (9) that

$$\omega = 1 - \Omega_0/[(1 + \varepsilon_x)(1 + \varepsilon_y) + 0,5\Omega_0\Omega_z] \\ (\Omega_0 = V_0^*/(2l_{x,0}l_{y,0}h_0)). \tag{10}$$

Equation (10) makes it possible to study the change in porosity during deformation.

It is impossible to find a formula that explicitly expresses the dependence of the strains ε_x, ε_y on the acting stress σ from Eqs. (5), (7), and (8) (excluding σ_x, σ_y, σ_z, ε_z, α), since Eqs. (5) and (7) are nonlinear, regardless of the physical relations for the elements of the composite. Thus, we will devise an algorithm for numerical computation which employs the method of "sequential loadings." This allows us to have a linear system of equations in increments of the unknowns for each load step.

We assume that the running vlaue of the acting stresses σ_k can be represented in the form

$$\sigma_k = \sigma_{k-1} + \Delta_k\sigma \quad \forall k = 1, 2, \ldots, \sigma_0 = 0 \tag{11}$$

and that the stresses and strains in the elements of the composite are as follows:

$$\sigma_{x,k} = \sigma_{x,k-1} + \Delta_k\sigma_x, \quad \varepsilon_{x,k} = \varepsilon_{x,k-1} + \Delta_k\varepsilon_x, \\ \forall k = 1, 2, \ldots, \quad \sigma_{x,0} = \varepsilon_{x,0} = 0. \tag{12}$$

We have similar relations for $\sigma_{y,k}$, $\sigma_{z,k}$, $\varepsilon_{y,k}$, $\varepsilon_{z,k}$, if we replace the subscript x in (12) by y and z, respectively. We will henceforth write out only the relations with the subscript x for such cases, indicating in parentheses that they are also valid when x is replaced by y and z. Also, for the angle of reinforcement α we obtain

$$\alpha_k = \alpha_{k-1} + \Delta_k\alpha \quad \forall k = 1, 2, \ldots \tag{13}$$

with a change in the acting stress on $\Delta_k\sigma$.

Inserting (11)-(13) into Eqs. (5), (7) and ignoring quantities of second-order smallness relative to the increments of the above-indicated quantities, we find that

$$(1 - \Omega_x)\Delta_k\sigma = \Omega_y\Delta_k\sigma_y - \Omega_x\left(\Delta_k\sigma_x - \frac{4\sigma_{x,k-1}\Delta_k\alpha}{\sin 2\alpha_{k-1}}\right)\operatorname{ctg}^2\alpha_{k-1}, \\ \Omega_x\Delta_k\sigma_x + \Omega_z(\Delta_k\sigma_z \sin^2\alpha_{k-1} + \sigma_{z,k-1}\Delta_k\alpha \sin 2\alpha_{k-1}) = 0; \tag{14}$$

$$\lambda_{z,k-1}\Delta_k\varepsilon_z = \lambda_{x,k-1}\Delta_k\varepsilon_x \sin^2\alpha_0 + \lambda_{y,k-1}\Delta_k\varepsilon_y \cos^2\alpha_0, \\ \Delta_k\alpha(\lambda_{y,k-1} + \lambda_{x,k-1}\operatorname{tg}\alpha_0 \operatorname{tg}\alpha_{k-1}) + \Delta_k\varepsilon_y \operatorname{tg}\alpha_{k-1} - \\ - \Delta_k\varepsilon_x \operatorname{tg}\alpha_0 = 0, \tag{15}$$

where $\lambda_{x,k-1} = 1 + \varepsilon_{x,k-1}$ $(x \to y, z)$.

We obtain a closed system of equations relative to $\Delta_k\sigma_x$, $\Delta_k\sigma_x$ $(x \to y, z)$ and $\Delta_k\alpha$ if we augment Eqs. (14)-(15) by physical relations for the elements of the composite written in increments. With nontrivial n_x, n_y, n_z, we find from (8) that

$$\text{at} \quad k = 1 \quad \Delta_1\sigma_x = B_x|\Delta_1\varepsilon_x|^{n_x-1}\Delta_1\varepsilon_x \; (x \to y, z); \tag{16}$$

$$\text{at } k = 2, 3, \ldots \quad \Delta_k\sigma_x = B_{x,k}\Delta_k\varepsilon_x \ (x \to y, z)$$
$$\left(B_{x,k} = B_x n_x |\varepsilon_{x,\,k-1}|^{n_x-1}\right). \tag{17}$$

In the case $n_x = n_y = n_z = 1$, Eqs. (17) are valid $\forall k = 1, 2, 3, \ldots$

Thus, given $\Delta_k\sigma$, we can readily use system (14)-(17) $\forall k = 1, 2, \ldots$ to determine $\Delta_k\sigma_x$, $\Delta_k\varepsilon_x$ ($x \to y, z$), and $\Delta_k\alpha$. Here, for k = 1 we generally find nonlinear system (14)-(16). This system is not difficult to solve if $\Delta_1\varepsilon_x$ is assigned in the first step and $\Delta_1\sigma$ is determined. With k = 2, 3, ..., algebraic system (14), (15), (17) will be linear relative to the unknown increments $\Delta_k\sigma_x$, $\Delta_k\varepsilon_x$ ($x \to y, z$), and $\Delta_k\alpha$. This system can be solved by any one of a number of well-known methods.

In the numerical calculation, it is necessary to choose $\Delta_k\sigma$ on the basis of the condition

$$\max\{|\Delta_k\sigma_x|, \ |\Delta_k\varepsilon_x| \ (x \to y, z), \ |\Delta_k\alpha|/\pi\} < \delta. \tag{18}$$

Here, $0 < \delta \ll 1$ and is determined by the condition that quantities of second-order smallness relative to the increments being examined can be ignored. Thus, in the general case, at each loading step it is necessary to correct $\Delta_k\sigma$ in accordance with (18). Here, we increase the parameter k until σ_k reaches the required value of σ.

Figures 2 and 3 show results of numerical calculations performed with the parameters $\delta = 0.01$, $n_z = 1$, $\alpha_0 = \pi/3$, $\Omega_0 = 0.2$, $\Omega_x = \Omega_y = 0.1$, $B_z/B_y = 20$ and

$$1) \ n_x = n_y = 2, \ B_x/B_y = 1; \ 2) \ n_x = n_y = 1, \ B_x/B_y = 1; \quad 3) \ n_x = n_y = 1, \ B_x/B_y = 3\ . \tag{19}$$

The solid curves correspond to the dependences of the longitudinal ε_y and transverse $|\varepsilon_x|$ strains on the acting stress σ/B_y. The dashed curves in Fig. 2 correspond to the dependence of the angle of reinforcement α on σ/B_y, and Fig. 3 characterizes the change in porosity ω in relation to σ/B_y. The numbers next to the curves represent the variant of the parameter from (19).

The results obtained here show that the proposed model of a composite material makes it possible to consider the nonlinear character of the $\sigma \sim \varepsilon_y$ and $\sigma \sim \varepsilon_x$ curves (even in the case of elastic deformation of the substructural elements - solid lines 2 and 3 in Figs. 2 and 3), determine the changes in the structure of the material (angle of reinforcement and porosity) during loading, and determine the transverse strains during uniaxial loading (here, as shown by the calculations, the ratio of the transverse strain to the longitudinal strain for the given materials depends on the acting stress).

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